EQUIVALENT INVARIANT MEASURES

BY

JOSEPH MAX ROSENBLATT

ABSTRACT

Let G be a finitely-generated group of non-singular measurable transformations of a measure space (X, β, p). Fix $A \in \beta$ with $p(A) > 0$. A general technique for groups gives sufficient conditions for there to exist a G-invariant measure ν equivalent to p with $\nu(A) = 1$. These conditions are phrased in terms of the growth behavior of $g \to p(gB)$ for $B \in \beta$. The question of necessity is handled in some special cases.

Let (X, β, p) be a σ-finite positive measure space. An *invertible measurable transformation* T of X is a one-to-one onto map $T: X \to X$ such that both T and T^{-1} are β-measurable. T is *non-singular* if $p(TE) = 0$ exactly when $p(E) = 0$ for $E \in \beta$. If T is non-singular then so is T^{-1}.

A *finitely-additive measure* u on (X, β) is a set function $u: \beta \to [0, \infty]$ such that $u(A \cup B) = u(A) + u(B)$ if A and B are disjoint measurable sets. Given two finitely-additive measures u and v on (X, β), we say u is *equivalent* to v when $u(E) = 0$ if and only if $v(E) = 0$ for $E \in \beta$. The term *measure* will be reserved for a countably-additive $u: \beta \to [0, \infty]$.

If u is a finitely-additive measure on (X, β) and T is an invertible measurable transformation of X, we say u is *T-invariant* when $u(TE) = u(E)$ for all $E \in \beta$. If G is a group of invertible measurable transformations of X under composition, u is G-invariant when u is T-invariant for all $T \in G$.

If $B \in \beta$, 1_B denotes the *characteristic function* of B. If F is a finite subset of a set X and f is a real-valued function of X then $\langle 1_F, f \rangle$ denotes $\sum_{x \in F} f(x)$. For any linear functional ϕ on a linear space E of real-valued functions on X, we write $\langle \phi, f \rangle$ for the value $\phi(f)$ when $f \in E$.

Received February 8, 1973 and in revised form June 18, 1973

Cardinality of a set F will be denoted by $\| F \|$. For sets A and B, $A \setminus B$ is the difference set and $A \Delta B$ is $(A \setminus B) \cup (B \setminus A)$.

1.

Let G be a group and let $\mathscr{L}$ be the linear space under pointwise addition of real valued functions on G. For $H \in \mathscr{L}$ and $g \in G$, $R_g H \in \mathscr{L}$ is defined by $R_g H(x) = H(xg)$. For a positive $H \in \mathscr{L}$ with $H \neq 0$, we define B_H to be the subspace of $\mathscr{L}$ consisting of all functions K such that there exists $g_1, \cdots, g_n \in G$ and $r > 0$ with $|K| \leqq r \sum_{i=1}^{n} R_{g_i} H$. B_H is then invariant under the action of G on $\mathscr{L}$ in that $R_g K \in B_H$ for any $g \in G$ and $K \in B_H$.

A linear functional ϕ on B_H is positive when $\langle \phi, K \rangle \geqq 0$ for all $K \in B_H$ with $K \geqq 0$. We say ϕ is G-invariant if $\langle \phi, R_g K \rangle = \langle \phi, K \rangle$ for all $g \in G$ and $K \in B_H$. For a finite set $F \subset G$ we write 1_F for the positive linear functional given by $\langle 1_F, K \rangle = \sum_{x \in F} K(x)$ for $K \in \mathscr{L}$.

We assume that G is finitely-generated by a finite set F. We write F^N for the set of all products $f_1 \cdots f_N$ where f_i is in F, F^{-1}, or $\{e\}$ for all $i = 1, \cdots, N$.

THEOREM 1.1. *Let $H \in \mathscr{L}$ be a positive non-zero function. Assume*

$$\lim_N \langle 1_{F^N}, H \rangle^{1/N} = 1.$$

Then there is a positive G-invariant linear functional ϕ on B_H such that $\langle \phi, H \rangle = 1$. Also, there is a subnet $\{N_\gamma\}$ of the sequence $\{1, 2, 3, \cdots\}$ such that for all $K \in B_H$,

$$\lim_\gamma \langle 1_{F^{N_\gamma}}, K \rangle / \langle 1_{F^{N_\gamma}}, H \rangle = \langle \phi, K \rangle.$$

PROOF. For each finite set $F_0 \subset G$, there is $M \geqq 1$ with $F_0 \subset F^M$. Hence, if $F_1 = F^M$ then $F_1 \supset F_0$ and $\lim_N \langle 1_{F_1^N}, H \rangle^{1/N} = 1$. Let $a_n = \langle 1_{F_1^{2^n}}, H \rangle$. By choosing M larger to begin with we may assume $a_n > 0$ for all $n \geqq 1$. $1 \leqq \liminf_{n \to \infty} a_{n+1}/a_n \leqq \liminf_{n \to \infty} a_n^{1/n} = 1$. So there is a sequence of integers n_k such that

$$\lim_{k \to \infty} \langle 1_{F_1^{n_k+2}}, H \rangle / \langle 1_{F_1^{n_k}}, H \rangle = 1.$$

It follows that for any $f \in F_0$,

$$\langle 1_{F_1^{n_k+1} \Delta F_1^{n_k+1} f}, H \rangle / \langle 1_{F_1^{n_k+1}}, H \rangle \leqq 2 \langle 1_{F_1^{n_k+2} \setminus F_1^{n_k}}, H \rangle / \langle 1_{F_1^{n_k}}, H \rangle \to 0.$$

Define an index set $\mathscr{A} = \{(F_0, M, \varepsilon)\}$ where F_0 is a finite subset of G, $M \geqq 1$, and $\varepsilon > 0$ for all members of $\mathscr{A}$. We order this set by $(F_0', M', \varepsilon') \geqq (F_0, M, \varepsilon)$ if and only if $F_0 \supset F_0'$, $M' \geqq M$, and $\varepsilon' \leqq \varepsilon$. $\mathscr{A}$ is thus a directed set. We choose $N = N(F_0, M, \varepsilon)$ such that $N \geqq M$ and for all $f \in F_0$

$$\langle 1_{F^N \Delta F^N f}, H\rangle / \langle 1_{F^N}, H\rangle < \varepsilon.$$

This gives us a net $\{N_\alpha \mid \alpha \in \mathscr{A}\}$ of integers $N_\alpha \geqq 1$ such that for all $g \in G$,

$$\lim_\alpha \langle 1_{F^{N_\alpha} \Delta F^{N_\alpha} g}, H\rangle / \langle 1_{F^{N_\alpha}}, H\rangle = 0.$$

Also, for all $N \geqq 1$ eventually $N_\alpha \geqq N$. Thus, $\{N_\alpha\}$ is a subnet of $\{1, 2, 3, \cdots\}$.

For simplicity write F_α for F^{N_α}. We know for all $g \in G$,

$$\left| 1 - \frac{\langle 1_{F_\alpha}, R_g H\rangle}{\langle l_{F_\alpha}, H\rangle} \right| = \left| \frac{\langle 1_{F_\alpha} - 1_{F_\alpha g}, H\rangle}{\langle 1_{F_\alpha}, H\rangle} \right| \leqq \frac{\langle 1_{F \Delta F_\alpha g}, H\rangle}{\langle 1_{F_\alpha}, H\rangle}.$$

So $\lim_\alpha \langle 1_{F_\alpha}, R_g H\rangle / \langle 1_{F_\alpha}, H\rangle = 1$, for all $g \in G$. For any $K \in B_H$ there are $g_1, \cdots, g_n \in G$ and $r > 0$ with $|K| \leqq r \sum_{i=1}^n R_{g_i} H$. Therefore, $|\langle 1_{F_\alpha}, K\rangle / \langle 1_{F_\alpha}, H\rangle| \leqq 2rn$ eventually. That is, the net of linear functionals $\phi_\alpha = 1_{F_\alpha} / \langle 1_{F_\alpha}, H\rangle$ is pointwise eventually bounded on B_H. This implies that some subnet ϕ_γ converges pointwise on B_H to a linear functional ϕ. It follows immediately that $\phi \geqq 0$ and $\langle \phi, H\rangle = 1$.

To see that ϕ is G-invariant let $g \in G$ and $K \in B_H$. Suppose $|K| \leqq r \sum_{i=1}^n R_{g_i} H$. Then

$$\begin{aligned}
&|\langle \phi, K\rangle - \langle \phi, R_g K\rangle| \\
&= \lim_\gamma \left| \frac{\langle 1_{F_\gamma}, K\rangle - \langle 1_{F_\gamma}, R_g K\rangle}{\langle 1_{F_\gamma}, H\rangle} \right| \\
&= \lim_\gamma \left| \frac{\langle 1_{F_\gamma}, K\rangle - \langle 1_{F_\gamma g}, K\rangle}{\langle 1_{F_\gamma}, H\rangle} \right| \\
&\leqq \lim_\gamma \frac{\langle 1_{F_\gamma \Delta F_\gamma g}, |K|\rangle}{\langle 1_{F_\gamma}, H\rangle} \\
&\leqq r \sum_{i=1}^n \lim_\gamma \frac{\langle 1_{F_\gamma \Delta F_\gamma g}, R_{g_i} H\rangle}{\langle 1_{F_\gamma}, H\rangle} \\
&= r \sum_{i=1}^n \lim_\gamma \frac{\langle 1_{F_\gamma g_i \Delta F_\gamma g g_i}, H\rangle}{\langle 1_{F_\gamma}, H\rangle}.
\end{aligned}$$

Since $1_{F_\gamma g_i \Delta F_\gamma g g_i} \leqq 1_{F_\gamma g_i \Delta F_\gamma} + 1_{F_\gamma \Delta F_\gamma g g_i}$, this last term converges to zero. Hence $\langle \phi, K\rangle = \langle \phi, R_g K\rangle$ for all $g \in G$ and $K \in B_H$. The linear functional ϕ and the net $\{N_\gamma\}$ satisfy our claim. ∎

LEMMA 1.2. *Let (X, β, p) be a σ-finite measure space. Assume we have a finitely-additive measure u which is equivalent to p. Let v be defined for $E \in \beta$ by*

$$v(E) = \inf\left\{ \sum_{i=1}^{\infty} u(E_i) : \bigcup_{i=1}^{\infty} E_i \supset E, E_i \in \beta \right\}.$$

Then v is a countably-additive measure and v is equivalent to p. If u is G-invariant for a group G of measurable transformations of (X,β) then v is G-invariant also.

PROOF. (See Calderón [1].)

REMARK. Calderón [1] is in the context where u is finite but his proof works equally well for any finitely-additive measure. Also, if we only know that $p(B) = 0$ implies $u(B) = 0$ for all $B \in \beta$, then $p(B) = 0$ implies $v(B) = 0$ for all $B \in \beta$.

THEOREM 1.3. *Let (X,β,p) be a σ-finite measure space. Let G be a group generated by a finite set F and suppose G acts as a group of nonsingular measurable transformations of (X,β,p). Assume we have a finitely-additive measure ψ on (X,β) with the following properties:*

(i) $\psi(B) = 0$ *if* $p(B) = 0$;

(ii) $\psi(gA) \neq 0$ *for some* $g \in G$;

(iii) $\lim_{N\to\infty} \left[\sum_{g\in F^N} \psi(gA)\right]^{1/N} = 1$;

(iv) *if* $p(B) > 0$ *then*

$$\liminf_{N\to\infty} \sum_{g\in F^N} \psi(gB) \Big/ \sum_{g\in F^N} \psi(gA) > 0.$$

Then there is a G-invariant measure v equivalent to p with $v(A) = 1$.

PROOF. Let $H(g) = \psi(gA)$ for all $g \in G$. It satisfies the conditions of (1.1), hence there is a G-invariant positive linear functional ϕ on B_H with $\langle\phi, H\rangle = 1$ and there is a subnet $\{N_\gamma\}$ of $\{1,2,3,\cdots\}$ such that $\langle 1_{F^{N_\gamma}}, K\rangle / \langle 1_{F^{N_\gamma}}, H\rangle \to \langle\phi, K\rangle$ for all $K \in B_H$.

For $B \in \beta$ define $p_B(g) = \psi(gB)$ for all $g \in G$. Define a finitely-additive measure u on (X,β) by

$$u(B) = \begin{cases} \langle\phi, p_B\rangle & \text{if } p_B \in B_H \\ \infty & \text{otherwise.} \end{cases}$$

Thus u is G-invariant since ϕ is G-invariant. If $p(B) = 0$ then $p(gB) = 0$ for all $g \in G$. So $p(B) = 0$ implies $\psi(gB) = 0$ for all $g \in G$ and $p_B = 0$. Hence $p(B) = 0$ implies $u(B) = 0$. If $p(B) > 0$ then either $u(B) = \infty$ or $u(B) = \langle\phi, p_B\rangle$ and

$p_B \in B_H$. In the second case, since (iv) holds and $\{N_\gamma\}$ gets arbitrarily large eventually, $u(B) > 0$.

Use Lemma 1.2 to obtain a G-invariant measure v equivalent to p from the finitely-additive measure u. Since $0 < v(A) \leqq u(A) = 1$, a constant multiple of v is the desired measure. ∎

COROLLARY 1.4. *Let G be a group of non-singular measurable transformations of a σ-finite measure space (X, β, p). Let G be generated by a finite set F. Let $A \in \beta$ with $p(A) > 0$. Assume*

$$\lim_{N\to\infty} \left[\sum_{g \in F^N} p(gA) \right]^{1/N} = 1.$$

Assume also that when $p(B) > 0$

$$\inf_{N \geqq 1} \sum_{g \in F^N} p(gB) \Big/ \sum_{g \in F^N} p(gA) > 0.$$

Then there exists a G-invariant measure v equivalent to p with $v(A) = 1$.

PROOF. Let $\psi(B) = p(B)$ for $B \in \beta$. Then ψ satisfies the four conditions of Theorem 1.3. ∎

REMARK. An interesting case in which Corollary 1.4 applies is when $p(X) = 1$ and G is a nilpotent group or at least contains a nilpotent subgroup of finite index. Then $\| F^N \|^{1/N} \to 1$ as $N \to \infty$; see Wolf [4]. It follows

$$\lim_{N\to\infty} \left[\sum_{g \in F^N} p(gA) \right]^{1/N} = 1.$$

So there is a G-invariant measure v equivalent to p with $v(A) = 1$ if for all $p(B) > 0$

$$\inf_{N \geqq 1} \sum_{g \in F^N} p(gB) \Big/ \sum_{g \in F^N} p(gA) > 0.$$

The following corollary also comes from Theorem 1.3.

COROLLARY 1.5. *Let G be a finitely-generated nilpotent group of non-singular measurable transformations of a σ-finite measure space (X, β, p). Assume there is a positive measurable function f on X with*

$$0 < \sup_{g \in G} \int_{gA} f\,dp < \infty.$$

Assume when $p(B) > 0$ then

$$\liminf_{N\to\infty} \sum_{g \in F^N} \int_{gB} f\,dp \Big/ \sum_{g \in F^N} \int_{gA} f\,dp > 0.$$

Then there exists a G-invariant measure v equivalent to p with $v(A) = 1$.

PROOF. Let $\psi(B) = \int_B f\,dp$ in Theorem 1.3. ■

REMARK. Clearly if p is equivalent to a σ-finite G-invariant measure v with $v(A) = 1$ then the function dv/dp satisfies Corollary 1.5. Corollary 1.4 is more interesting than 1.5 because it does not depend on finding the function f. Horowitz [3] has a result similar to Corollary 1.4 for an ergodic, conservative Markov operator.

If the $\sup_N \sum_{g\in F^N} p(gA) < \infty$, then the hypotheses of Corollary 1.4 are all satisfied. This is the dissipative case.

2.

It is an open question whether a converse of Corollary 1.4 holds. One should perhaps also assume in this context that $\bigcup_{g\in G} gA = X$ and

$$\lim_{N\to\infty}\left[\sum_{g\in F^N} p(gA)\right]^{1/N} = 1.$$

Then the question is, assuming there is a G-invariant measure v equivalent to p with $v(A) = 1$, does it follow that for all B with $p(B) > 0$

$$\inf_{N\geqq 1} \sum_{g\in F^N} p(gB) \Big/ \sum_{g\in F^N} p(gA) > 0\ ?$$

We say a subset $B\in\beta$ is A-bounded when $1_B \leqq \sum_{i=1}^n 1_{g_iA}$ a.e. $[p]$ for some $g_1,\cdots,g_n\in G$. Since $\bigcup_{g\in G} gA = X$, any B with $p(B) > 0$ there is an A-bounded $B_0 \subset B$ with $p(B_0) > 0$. So the converse need only be verified for A-bounded sets.

There is at least one case in which the above condition does hold. We say G is *ergodic* if all G-invariant sets $B\in\beta$ satisfy $p(B) = 0$ or $p(X\setminus B) = 0$. If G is ergodic and $p(B) > 0$ then $\bigcup_{g\in G} gB = X$ a.e. $[p]$.

PROPOSITION 2.1. *Assume (X,β,p) is a measure space with $A\in\beta$ such that,*

$$\lim_{N\to\infty}\left[\sum_{g\in F^N} p(gA)\right]^{1/N} = 1.$$

Assume G is ergodic and for all A-bounded $B\in\beta$

$$\lim_{N\to\infty} \sum_{g\in F^N} p(gB) \Big/ \sum_{g\in F^N} p(gA) \text{ exists.}$$

Then if $p(B) > 0$,

$$\inf_{N\geqq 1} \sum_{g\in F^N} p(gB) \Big/ \sum_{g\in F^N} p(gA) > 0.$$

PROOF. Let $u(B)$ be the

$$\lim_{N\to\infty} \sum_{g\in F^N} p(gB) \Big/ \sum_{g\in F^N} p(gA)$$

for A-bounded sets $B\in\beta$. Let $u(B) = \infty$ otherwise. Since

$$\lim_{N\to\infty} \left[\sum_{g\in F^N} p(gA)\right]^{1/N} = 1,$$

there is a sequence $\{N_i\}$ such that for all $h\in G$,

$$\lim_{i\to\infty} \left(\sum_{g\in F^{N_i}h\Delta F^{N_i}} p(gA)\Big/ \sum_{g\in F^{N_i}} p(gA)\right) = 0.$$

Using $\{N_i\}$, as in Theorem 1.1, shows u is G-invariant. Let v be induced by u as in Lemma 1.2. Then v is a G-invariant measure and $p(B) = 0$ implies $u(B) = 0$ and hence, $v(B) = 0$. But we also know by the Vitali-Hahn-Saks theorem that $u = v$ when restricted to any A-bounded set because there u is a measure. If $p(B) > 0$, G ergodic implies $\bigcup_{g\in G} gB = X$ a. e. $[p]$. Since $v(A) = 1$ and v is G-invariant, $v(B) > 0$. Hence, for any A-bounded B with $p(B) > 0$ we have $u(B) > 0$ and thus,

$$\inf_{N\geqq 1} \sum_{g\in F^N} p(gB)\Big/ \sum_{g\in F^N} p(gA) > 0. \qquad \blacksquare$$

REMARK. It is not clear that the limit $u(B)$ need exist for all A-bounded B when there exists a G-invariant measure v equivalent to p with $v(A) = 1$. This is also not clear in the case we replace the index N by N_i chosen as in the proof of (2.1). In the following we will see that such limiting behavior is the case when $v(X) < \infty$ and $p(X) < \infty$.

We consider now a finitely-generated group G and a fixed sequence of finite sets $F_N \subset G$ such that for all $g\in G$

$$\frac{\| gF_N \Delta F_N \|}{\| F_N \|} \to 0.$$

We also assume $F_N^{-1} = F_N$ for all N. Such a sequence $\{F_N\}$ exists if and only if G is amenable. (See [2] for the construction of such sequences.) We call $\{F_N\}$ a Følner sequence.

We assume we have a probability space (X, β, p) on which G acts as a group of non-singular measurable transformations. The measure gp is defined by $gp(E) = p(gE)$ for all $E\in\beta$. For any $g\in G$, gp is equivalent to p and there is a positive measurable function w_g such that $gp(E) = \int_E w_g(x)dp(x)$ for all $E\in\beta$.

Let $L_1(p)$ be the absolutely-integrable functions of X with $\|f\|_1 = \int_X |f|\,dp$.

Let $L_\infty(p)$ be the a.e. bounded functions with $\|f\|_\infty = \inf\{K: K \geqq |f| \text{ a.e. } [p]\}$. Define $L_gf(x) = f(gx)$ and $\tau_g f = L_g f w_g$ for all $f \in L_1(p)$ and $g \in G$. Then $\|\tau_g f\|_1 = \|f\|_1$ for all $f \in L_1(p)$. Let ϕ_N be the linear operator defined by $\phi_N f = (1/\|F_N\|)\sum_{g\in F_N}\tau_g f$. Then $\|\phi_N f\|_1 \leqq \|f\|_1$ for all $f \in L_1(p)$. The question we are concerned with here is the $\|\cdot\|_1$-convergence of $\phi_N f$ for $f \in L_1(p)$.

THEOREM 2.2. *The following statements are equivalent*:

(i) *For all* $f \in L_1(p)$ *there is* $f^* \in L_1(p)$ *such that* $\phi_N f \to f^*$ in *the* $\|\cdot\|_1$ *norm*.

(ii) *There is* $f_0 \in L_1(p)$ *such that* $1/(\|F_N\|)\sum_{g\in F_N} w_g \to f_0$ *in the* $\|\cdot\|_1$ *norm.*

PROOF. Since $\phi_N 1 = 1/(\|F_N\|)\sum_{g\in F_N} w_g$, (i) implies (ii). Assume (ii). Let $S_1 = \{\tau_g f - f : f \in L_1(p) \text{ and } g \in G\}$ and $S_2 = \{f \in L_\infty(p): L_g f = f \text{ for all } g \in G\}$. Let $S_0 = \text{Span } S_1 \cup S_2$. For $f \in L_1(p)$

$$\|\phi_N(\tau_g f - f)\|_1 = \frac{1}{\|F_N\|}\Big\|\sum_{h\in F_N}(\tau_h\tau_g f - \tau_h f)\Big\|^1$$

$$\leqq \frac{1}{\|F_N\|}\sum_{h\in gF_N\Delta F_N}\|\tau_h f\|_1$$

$$\leqq \|f\|_1\frac{\|gF_N\Delta F_N\|}{\|F_N\|}\to 0.$$

So $\phi_N S$ converges to zero in the norm $\|\cdot\|_1$ for $S \in S_1$. For $S \in S_2$, $\phi_N S = S\sum_{g\in F_N} w_g/\|F_N\| = S\phi_N 1$; so $\|\phi_N S - Sf_0\|_1 \leqq \|S\|_\infty\|\phi_N 1 - f_0\|_1$ where f_0 is as in (ii). Hence, $\phi_N S \to Sf_0$ in the $\|\cdot\|_1$ norm for all $S \in S_2$. We have for all $S \in S_0$, there exists $S^* \in L_1(p)$ such that $\phi_N S \to S^*$ in the $\|\cdot\|_1$ norm. Since the operators ϕ_N are uniformly bounded, if we show S_0 is $\|\cdot\|_1$-dense in $L_1(p)$ then we are finished. But if S_0 is not $\|\cdot\|_1$-dense there exists $H \in L_\infty(p)$ such that $H \neq 0$ and $0 = \int SH\,dp$ for all $S \in S_0$. But then for all $f \in L_1(p)$,

$$0 = \int(\tau_g f - f)H\,dp = \int f(L_{g^{-1}}H - H)dp.$$

This implies $L_g H = H$ for all $g \in G$. Hence, $H \in S_2$ and $0 = \int H^2 dp$. So $H = 0$, a contradiction. ∎

REMARK. If p is invariant, each $w_g = 1$ and (ii) is trivially satisfied. Thus (i) holds if p is invariant. This result is in Greenleaf [2].

The following corollary will further clarify the condition of Theorem 2.2.

COROLLARY 2.3. *Let* (X,β,p) *and* G *be as above. Consider the following properties of* (X,β,p) *and* G:

(i) *There exists a finite G-invariant measure equivalent to p.*

(ii) *For all* $f \in L_1(p)$, $\phi_N f$ *converges in* $L_1(p)$.

(iii) *For all* $B \in \beta$, $(1/\| F_N \|) \sum_{g \in F_N} p(gB)$ *converges.*

(i) *implies* (ii) *and* (ii) *implies* (iii) *always. If G is ergodic then* (iii) *implies* (i).

PROOF. Assume v is a finite G-invariant measure equivalent to p. Let $f_0 \in L_1(v)$ with $f_0 \geqq 0$ such that $p(E) = \int_E f_0 dv$ for all $E \in \beta$. By Theorem 2.2 applied to (X, β, v) and G, there exists $f_0^* \in L_1(v)$ such that

$$\int \left| \frac{1}{\| F_N \|} \sum_{g \in F_N} L_g f_0 - f_0^* \right| dv \to 0.$$

Then $f_0^*/f_0 \in L_1(p)$ and for all $g \in G$, $L_g f_0 = w_g f_0$. Therefore

$$\int \left| \frac{1}{\| F_N \|} \sum_{g \in F_N} w_g - f_0^*/f_0 \right| dp$$

$$= \int \left| \frac{1}{\| F_N \|} \sum_{g \in F_N} L_g f_0 - f_0^* \right| \frac{1}{f_0} dp$$

$$= \int \left| \frac{1}{\| F_N \|} \sum_{g \in F_N} L_g f_0 - f_0^* \right| dv \to 0.$$

Hence, by Theorem 2.2, we have (i) implies (ii). If (ii) holds, then in particular there is $f_1 \in L_1(p)$ such that $\phi_N 1 \to f_1$ in the $\| \cdot \|_1$ norm. Since

$$\left| \int_B \frac{1}{\| F_N \|} \sum_{g \in F_N} w_g dp - \int_B f_1 dp \right|$$

$$\leqq \left\| \frac{1}{\| F_N \|} \sum_{g \in F_N} w_g - f_1 \right\|_1 \to 0,$$

$\lim_{N \to \infty} \int_B (1/\| F_N \|) \sum_{g \in F_N} w_g) dp$ exists. This is (iii). If (iii) holds then $v(B) = \lim_{N \to \infty} (1/\| F_N \|) \sum_{g \in F_N} p(gB)$ defines a probability measure by the Vitali-Hahn-Saks theorem. Because of the conditions on the sequence $\{F_N\}$ v is G-invariant. If $p(B) = 0$, then each $p(gB) = 0$ and $v(B) = 0$. Now if G is ergodic then $p(B) > 0$ implies $\bigcup_{g \in G} gB = X$. a.e. $[p]$ So $v(B) > 0$ as well. Thus v is a measure as in (i). ■

REMARK. If (iii) holds then for any $\varepsilon > 0$ there is a G-invariant $B \in \beta$ with $p(X/B) < \varepsilon$ and a finite measure v supported on B such that v is equivalent to p on B and v is G-invariant. Using this, one can show (iii) implies (i) always.

If case (i) holds, we actually have when $p(B) > 0$ then $\inf_{g \in G} p(gB) > 0$. Hence, the ratio condition of Corollary 1.4 holds and the condition that $\lim_N \| F^N \|^{1/N} = 1$ is the same as the condition that $\lim_N [\sum_{g \in F^N} p(gA)]^{1/N} = 1$. Also, if we choose $\{N_i\}$ as in Proposition 2.1 then $\{F^{N_i}\}$ is a Følner sequence and Corollary 2.3 (iii) holds. This should be compared with the remark after Proposition 2.1.

References

1. A. Calderón, *Sur les mesures invariants*, C. R. Acad. Sci. Paris **240** (1955), 1960–1962.

2. F. Greenleaf, *Ergodic theorems and the construction of summing sequences in amenable locally compact groups* (to appear in Comm. Pure and Appl. Math).

3. S. Horowitz, *On σ-finite invariant measures for Markov processes*, Israel J. Math. **6** (1968), 338–345.

4. J. Wolf, *Growth of finitely-generated solvable groups*, J. Differential Geometry (4) **1** (1968), 421–446.

University of British Columbia
Vancouver, Canada